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FINITELY GENERATED COHOMOLOGY HOPF ALGEBRAS

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§1

WE WORK in the category whose objects are topological spaces which have the homotopy type of CW-complexes with finite skeletons, and whose morphisms are continuous maps. The space X is an H-space if there is a continuous multiplication

$$m : X \times X \to X$$

with unit.

Let X be a connected H-space and suppose that its integral cohomology, $H^*(X, Z)$, is finitely generated as a group. A classical theorem of Hopf implies that the rational cohomology ring, $H^*(X, Q)$, is an exterior algebra on odd dimensional generators. The purpose of this note is to strengthen this result.

THEOREM 1.1. *Let X be 2-connected and an H-space with $H^*(X, Z)$ finitely generated as a ring, then $H^*(X, Q)$ is an exterior algebra on odd dimensional generators.*

If $H^*(X, Z)$ is not finitely generated as a ring, in general the conclusion of Theorem 1.1 would clearly be false, and the existence of $K(Z, 2)$, infinite dimensional complex projective space, whose integral cohomology ring is a polynomial algebra on a single generator of dimension 2, implies the necessity of some condition such as X being 2-connected. In fact, as will become clear in the proof, both the conditions on the H-space X may be weakened. The condition that X is 2-connected may be replaced by the two conditions:

(1.1) X is connected,

(1.2) $H^2(X, Q) \subset H^1(X, Q) \cdot H^1(X, Q)$, under cup product,

and the condition that the ring $H^*(X, Z)$ is finitely generated may be replaced by the two conditions,

(1.3) $H^*(X, Z)$ has no p-torsion for some prime integer p,

(1.4) A homogeneous minimal set of ring generators for $H^*(X, Z_p)$ has only a finite number of generators of even dimension.

There are two main ingredients for the proof of Theorem 1.1. These are Theorem 1.2 and Lemma 1.3. As usual p is a prime integer and Z_p is the field of integers reduced modulo p.

THEOREM 1.2. *Let X be a connected H-space and assume that*:

(1) *$H^*(X, Z)$ had no p-torsion,*

(2) *$H^*(X, Z_p)$ is finitely generated as a ring,*

(3) *$H^*(X, Z_p)$ as an algebra, is the tensor product of an exterior algebra and a polynomial algebra,*

then the generators of the polynomial algebra all have dimension 2.

Let Q_p be the ring of rational numbers with denominators prime to p.

LEMMA 1.3. *Let Y be any space with $H^*(Y, Z)$ free of p-torsion. Suppose that $x \in H^n(Y, Q_p)$, $n > 0$, and $x^p = py$ for some $y \in H^{pn}(Y, Q_p)$, then $y^p = pz$ for some $z \in H^{p^2n}(Y, Q_p)$.*

This present note is a bi-product of a longer paper on even dimensional cohomology Hopf algebras which the author is preparing, but the technicalities of this longer paper are not necessary for the proof of Theorem 1.1 and so it seemed desirable to present this simplified version. The methods of proof are K-theoretic and the proof of Theorem 1.2 requires familiarity with [6]. In §2 we recall come of the K-theoretic properties which are needed and prove Lemma 1.3, and in §3 the proofs of Theorems 1.2 and 1.1 are given.

§2

Consider complex K-theory with Q_p coefficients for finite complexes, that is, take the tensor product of the integral K-theory of [3] with Q_p. If a space is not a complex with finite skeletons, we replace it with such a complex of the same homotopy type. Then the nature of our arguments is such that it is always sufficient to consider a skeleton of sufficiently high dimension.†

Let X be a finite complex with $H^*(X, Z)$ free of p-torsion. The Atiyah–Hirzebruch spectral sequence of [3], with Q_p coefficients, collapses, since $H^*(X, Q_p)$ is torsion free. Thus

$$H^n(X, Q_p) \cong K_n^*(X, Q_p)/K_{n+1}^*(X, Q_p) \tag{2.1}$$

and the isomorphism preserves the natural ring structures which each side possesses. We shall write $H^*(X)$ and $K^*(X)$ for $H^*(X, Q_p)$ and $K^*(X, Q_p)$, then $K^*(X) = K^\circ(X) \oplus K^1(X)$ and (2.1) may now be written as (2.2)

$$H^{2n}(X) \cong K^\circ_{2n}(X)/K^\circ_{2n+1}(X),\ H^{2n+1}(X) \cong K^1_{2n+1}(X)/K^1_{2n+2}(X). \tag{2.2}$$

We shall also use the Adams operators ψ^k of [1]. In the proof of Lemma 1.3 we need the following three properties of ψ^p acting on $K^\circ(X)$.

(2.3) ψ^p is a ring homomorphism,

(2.4) $\psi^p(u) = u^p \bmod p$, for all $u \in K^\circ(X)$,

(2.5) Suppose that $H^*(X, Z)$ is free of p-torsion and let $u \in K^\circ_{2n}(X)$, then there exist $n+1$ elements u_i, $0 \le i \le n$, such that $u_i \in K^0_{2n+2i(p-1)}(X)$ and

$$\psi^p(u) = \sum_{0 \le i \le n} p^{n-i} u_i .$$

Proofs of (2.3) and (2.4) can be found in [1]. (2.5) follows almost immediately from Theorem 5.6 and Lemma 5.5 of [2].

† We do not consider the inverse limit K-theory over the finite skeletons because of difficulties with the coefficients, as $\varprojlim$ does not commute with tensor products.

Proof of Lemma 1.3. If $x \in H^n(X)$ and n is odd, the result is trivial since $x^2 = 0$, $y = 0$, and so we may take $z = 0$. Therefore let $x \in H^{2m}(X)$ and choose $u \in K^{\circ}_{2m}(X)$ whose image in $H^{2m}(X)$ under the isomorphism (2.2) is x. Since $x^p = py$,

$$u^p = pv + w \tag{2.6†}$$

where the image of v in $H^{2pm}(X)$ is y and $w \in K^{\circ}_{2pm+2}(X)$. Thus (2.4) implies that

$$\psi^p(u) = pw_1 + w$$

for some $w_1 \in K^{\circ}(X)$, and since ψ^p is a ring homomorphism (2.3),

$$\psi^p(u^p) = (pw_1 + w)^p, \text{ or}$$

$$\psi^p(u^p) = p^2 w_2 + w^p \tag{2.7}$$

where $w_2 \in K^{\circ}(X)$. Combining (2.6) and (2.7)

$$\psi^p(pv + w) = p^2 w_2 + w^p. \tag{2.8}$$

Now $w \in K^{\circ}_{2pm+2}(X)$ and so $w^p \in K^{\circ}_{2p^2m+2}(X)$, and (2.5) implies that

$$\psi^p(w) = p^2 w_3 \bmod K^{\circ}_{2p^2m+2}(X)$$

for some w_3. Thus (2.8) implies that

$$\psi^p(pv) = p^2 w_4 \bmod K^{\circ}_{2p^2m+2}(X),$$

and as there is no torsion

$$\psi^p(v) = pw_4 \bmod K^{\circ}_{2p^2m+2}(X).$$

But $\psi^p(v) = v^p \bmod p$, by (2.4), and so

$$v^p = pw_5 \bmod K^{\circ}_{2p^2m+2}(X), \text{ for some } w_5. \tag{2.9}$$

Now if $y^p = 0$, the conclusion of the lemma is trivially satisfied with $z = 0$, and if $y^p \neq 0$, the image of v^p in $H^{2p^2m}(X)$ under the isomorphism (2.2) coincides with y^p. Thus if z is the image of w_5 in $H^{2p^2m}(X)$ under (2.2), (2.9) implies that $y^p = pz$ which completes the proof of the lemma.

§3

We need the following definition. A is a quasi-monogenic Hopf algebra over an integral domain I, if $A \otimes F$ is a monogenic Hopf algebra, where F is the field of fractions of I.

Let X be a connected H-space and assume that $H^*(X, Z)$ is free of p-torsion. The following theorem gives the ring structure in the cohomology with Q_p coefficients.

THEOREM 3.1. (*Clark*) *$H^*(X, Q_p)$ as an algebra is the tensor product of quasi-monogenic Hopf algebras.*

Proof. This is a particular case of Theorem 1.6 of [5]. Alternatively a more elementary proof (only applicable to cohomology Hopf algebras) can be given based on Lemma 1.3

†It is not true in general that we may choose v so that $w = 0$. A counter example for $p = 3$ occurs with $X = \Omega E_8$, the loop space on the exceptional Lie group E_8, with x an indecomposable element in $H^{14}(\Omega E_8, Q_3)$.

and the corresponding theorem of Borel for a Hopf algebra over a field of characteristic $p > 0$, Theorem 6.1 of [4].

There are three types of quasi-monogenic Hopf algebras A which can occur in Theorem 3.1, which we call A1, A2 and A3.

(A.1) A is an exterior algebra on a single odd dimensional generator,

(A.2) A is a polynomial algebra on a single even dimensional generator,

(A.3) A has an infinite sequence of even dimensional generators,

$$1, x_1, x_r, x_{r+1}, \ldots, x_n, x_{n+1}, \ldots$$

where $x_1^{p^{r-1}} = x_r$ and $x_{r+i}^p = px_{r+i+1}$, all $i \geq 0$, and there are no other relations,

The verification, using Lemma 1.3, is straightforward. Alternatively it follows from a theorem of Halpern (see for example Theorem 2.1 of [5]) and Lemma 1.3.

Proof of Theorem 1.2. The central point of the proof is to apply Theorem 3.1 of [6]. Theorem 3.1 above and the hypotheses of Theorem 1.2 imply that $H^*(X)$ as an algebra is the tensor product of a finite number of quasi-monogenic Hopf algebras of types A1 and A2. It follows from the isomorphism (2.1) that $K^*(X)$† is isomorphic as a ring to $H^{**}(X)$. and therefore is the tensor product of an exterior algebra and a polynomial algebra. Let $T = T^\circ + T^1$ be the ideal in $K^*(X)$ generated by $K^1(X)$. It is clear for dimensional reasons that T is a Hopf ideal in the Z_2 Hopf algebra $K^*(X)$†. Thus

$$K^*(X) = T \oplus M$$

where M is a polynomial algebra which lies in $K^\circ(X)$. Further we may choose generators for M, u_i $1 \leq i \leq m$ say, so that if u_i has exact filtration $2q$ in $K^\circ(X)$, the images of all the u_i in $H^{2q}(X)$ under the isomorphism (2.2) generate a polynomial subalgebra of $H^*(X)$ which is a direct complement of the ideal in $H^*(X)$ generated by $H^{\text{odd}}(X)$.

LEMMA 3.2. $\psi^k(T^\circ) \subset T^\circ$, *for all* K.

Proof. Let $u \in K^\circ(X)$, then $u \in T^\circ$ if and only if $u^2 = 0$. The result follows as ψ^k is a ring homomorphism.

Now let $f : X \to X$ be the H-space p-th power map on X with respect to some fixed order of multiplication. For example if

$$\Delta : X \to X \times X \qquad \text{and} \qquad m : X \times X \to X$$

are the diagonal map and the H-space multiplication map respectively, set

$$f = m(1 \times m)(1 \times 1 \times m) \cdots (1 \times 1 \times \Delta)(1 \times \Delta)\Delta \qquad (2p-2 \text{ factors}).$$

If necessary, we alter f by a homotopy to ensure that it is cellular.

LEMMA 3.3. $f^!(T^\circ) \subset T^\circ$.

Proof. T is a sub Hopf algebra of $K^*(X)$ and so $f^!(T) \subset T$.

† Strictly speaking, as we have replaced X in the K-theory by a finite skeleton, these statements are only true modulo elements of high filtration, but this is sufficient for our purposes.

We now use the notations and results of [6]. First we replace X by a finite skeleton of large dimension whose cohomology is free of p-torsion. We may still assume that T is an ideal in $K^*(X)$ and Lemma 3.2 and Lemma 3.3 are true. T^0 is an ideal and a direct summand in $K^0(X)$ and Lemma 3.2 implies that M is a multiplicative ψ^k-module over Q_p, using Lemma 2.2 of [6]. The discussion at the beginning of §3 of [6] implies that we may suppose that M is a truncated polynomial algebra of height $p+1$. The graded ring corresponding to M, N say, is also truncated polynomial of height $p+1$. Further f induces on M a p-map in the sense of §3 of [6], using Lemma 3.3 above and Lemma 2.10 of [6].

It follows from Theorem 3.1 of [6] that all generators of N have dimension 1 corresponding to cohomological dimension 2 and the proof of Theorem 1.2 is completed.

Proof of Theorem 1.1. We need to show that there is some prime p for which $H^*(X, Z)$ has no p-torsion so that we may apply the earlier results.

Let $i : H^*(X, Z) \to H^*(X, Q)$ be induced by coefficient inclusion $Z \subset Q$. Let $Q(i) : Q\{H^*(X, Z)\} \to Q\{H^*(X, Q)\}$ be the homomorphism induced between the quotient modules of indecomposable elements. Choose a homogeneous base for the finitely generated Abelian group $Q\{H^*(X, Z)\} = F \oplus T$ such that x_i', $1 \le i \le n$, is a basis for the free group F and y_j', $1 \le j \le m$, is a basis for the torsion group T, with $k_j y_j' = 0$ for each j. The elements $Q(i)(x_i')$, $1 \le i \le n$, form a basis for $Q\{H^*(X, Q)\}$ over Q. Let x_i and y_j be homogeneous representatives in $H^*(X, Z)$ for x_i' and y_j'.

Let $K = k_1 k_2, \ldots, k_m$ and express $2K$ as a product of powers of distinct primes.

LEMMA 3.4. *$H^*(X, Z)$ can have torsion only for those primes which occur in the prime power decomposition of $2K$.*

The proof of Lemma 3.4 will require two subsidiary lemmas. In the first we use the fact that X is an H-space.

LEMMA 3.5. *Let $p(x_1, x_2, \ldots, x_n)$ be some polynomial expression over Z in the x_i such that $i\{p(x_1, x_2, \ldots, x_n)\} = 0$ in $H^*(X, Q)$; then $2p(x_1, x_2, \ldots, x_n) = 0$ in $H^*(X, Z)$.*

Proof. The Leray structure theorem for an associative, commutative Hopf algebra over a field of characteristic zero (see for example, Theorem 7.5 of [7]), implies that the algebra $H^*(X, Q)$ is free commutative over Q. The elements $i(x_i)$, $1 \le i \le n$, form a multiplicative basis for $H^*(X, Q)$ over Q by the remarks above. Thus if $p(i(x_1), i(x_2), \ldots, i(x_n)) = 0$; then the polynomial $p(x_1, x_2, \ldots, x_n)$ is either identically zero or is a sum of monomals in the x_i in each of which the exponent of some odd dimensional x_i is greater than one. Thus $2p(x_1, x_2, \ldots, x_n) = 0$.

LEMMA 3.6. *Let $w \in H^q(X, Z)$; then $K^{mq} w = p(x_1, x_2, \ldots, x_n)$, some polynomial expression in the x_i.*

Proof. Assume that the y_j are ordered so that the dimensions do not decrease. Let $u = f(x_1, x_2, \ldots, x_n, y_1, \ldots, y_s)$ be any polynomial expression in $H^q(X, Z)$, for some s in $1 \le s \le m$. We shall show that $K^q u = g(x_1, x_2, \ldots, x_n, y_1, \ldots, y_{s-1})$, which by repeated application implies the conclusion of the lemma. In fact

$$K^q u = K^q f(x_1, x_2, \ldots, x_n, y_1, \ldots, y_s)$$
$$= g(x_1, x_2, \ldots, x_n, k_1 y_1, \ldots, k_s y_s),$$

for dimensional reasons, where again g is some polynomial expression. But $k_i y_i$ is decomposable in terms of elements of lower dimension, or is zero, and so

$$K^q u = h(x_1, x_2, \ldots, x_n, y_1, \ldots, y_{s-1})$$

for some polynomial h. This completes the proof of the lemma.

The proof of Lemma 3.4 is now clear. Let w be a torsion element in $H^q(X, Z)$. Then by combining Lemma 3.6 and Lemma 3.5, we see that $2K^{mq}w = 0$. The Euclidean algorithm then implies that if $pw = 0$ for some prime p; then the prime p occurs in the prime power decomposition of $2K$. This is sufficient to prove Lemma 3.4.

We complete the proof of Theorem 1.1 as follows. Using Lemma 3.4, choose p such that $H^*(X, Z)$ has no p-torsion. Coefficient inclusion $Z \subset Q_p$ induces a ring homomorphism $j: H^*(X, Z) \to H^*(X)$, and also a linear map $Q(j): Q\{H^*(X, Z)\} \to Q\{H^*(X)\}$ between the indecomposable quotients. Now $H^*(X) = H^*(X, Z) \otimes Q_p$, by the universal coefficient theorem, and so if $u \in H^*(X)$, there exists a unit $\lambda \in Q_p$ such that $\lambda u \in j\{H^*(X, Z)\}$. Therefore $Q(j)\{Q\{H^*(X, Z)\}\} \otimes Q_p = Q\{H^*(X)\}$, and so $H^*(X)$ is finitely generated as an algebra over Q_p. Theorem 3.1 and the remarks which follow it then imply that the algebra $H^*(X)$ is a tensor product of a finite number of Hopf algebras of types A1 and A2, since any factor of type A3 is not finitely generated. Reducing modulo p, Theorem 1.2 implies that the generator of any factor of type A2 must have dimension 2. But $H^2(X) \cong H_2(X, Z) \otimes Q_p \cong \pi_2(X) \otimes Q_p = 0$, using the facts that $H^*(X)$ is free of p-torsion and that X is 2-connected. Thus there are no factors of type A2 and so $H^*(X)$, and therefore $H^*(X, Q)$, is an exterior algebra on odd dimensional generators.

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